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DEMONSTRATION OF A FUNDAMENTAL THEOREM OBTAINED BY MR. SYLVESTER.

BY B. LIPSCHITZ, *Professor Ordinarius in the University of Bonn.*

IN a memoir published in the 85th volume of Mr. Borchart's Journal, Mr. Sylvester has developed the thought of a new notation for algebraical forms or quantics. Being given any algebraical homogeneous function of the n variables $x_1, x_2, \ldots x_n$, whose degree is denoted by p, let the powers and products of powers of the said degree $x_1^p, x_1^{p-1}x_2 \ldots x_n^p$ be expressed by $X_1, X_2, \ldots X_\nu$, and the corresponding polynomial coefficients by $\pi_1, \pi_2, \ldots \pi_\nu$, then the proposed form may be expressed by applying as numerical factors to the constants $f_1, f_2, \ldots f_\nu$ the square roots of the respective polynomial coefficients, so that

$$(1) \qquad F(x_1, x_2, \ldots x_n) = \sqrt{\pi_1} f_1 X_1 + \sqrt{\pi_2} f_2 X_2 + \ldots + \sqrt{\pi_\nu} f_\nu X_\nu,$$

and is called in that shape a *prepared* form. Now suppose that on introducing instead of the n variables $x_1, x_2, \ldots x_n$ the n linear functions of n new variables $y_1, y_2, \ldots y_n$, such that

$$(2) \qquad x_a = k_{a,1} y_1 + k_{a,2} y_2 + \ldots + k_{a,n} y_n,$$

where the letter a runs through the numbers $1, 2, \ldots n$, the form $F(x_1, x_2, \ldots x_n)$ is changed into the form $G(y_1, y_2, \ldots y_n)$, written likewise as a prepared form, so that

$$(3) \qquad G(y_1, y_2, \ldots y_n) = \sqrt{\pi_1} g_1 Y_1 + \sqrt{\pi_2} g_2 Y_2 + \ldots + \sqrt{\pi_\nu} g_\nu Y_\nu.$$

It is always understood that in substituting for the letter x the letters y, z, t, u, the functions X_α are respectively turned into the functions $Y_\alpha, Z_\alpha, T_\alpha, U_\alpha$, the letter α going through the numbers $1, 2, \ldots \nu$. As the constant elements $g_1, g_2, \ldots g_\nu$ depend in a linear manner upon the elements $f_1, f_2, \ldots f_\nu$, we get by partial differentiation the ν equations

$$(4) \qquad \begin{aligned} g_1 &= \frac{\delta g_1}{\delta f_1} f_1 + \frac{\delta g_1}{\delta f_2} f_2 + \ldots + \frac{\delta g_1}{\delta f_\nu} f_\nu, \\ &\ldots\ldots\ldots\ldots\ldots, \\ &\ldots\ldots\ldots\ldots\ldots, \\ g_\nu &= \frac{\delta g_\nu}{\delta f_1} f_1 + \frac{\delta g_\nu}{\delta f_2} f_2 + \ldots + \frac{\delta g_\nu}{\delta f_\nu} f_\nu, \end{aligned}$$

whence Mr. Sylvester derives the expression, that the substitution operated on the variables

$$(5)\qquad \begin{array}{l} k_{1,1},\ k_{1,2},\ \ldots\ k_{1,n}, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad, \\ k_{n,1},\ k_{n,2},\ \ldots\ k_{n,n}, \end{array}$$

induces the substitution operated on the elements

$$(6)\qquad \begin{array}{l} \dfrac{\delta g_1}{\delta f_1},\ \dfrac{\delta g_1}{\delta f_2},\ \ldots\ \dfrac{\delta g_1}{\delta f_\nu}, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad, \\ \dfrac{\delta g_\nu}{\delta f_1},\ \dfrac{\delta g_\nu}{\delta f_2},\ \ldots\ \dfrac{\delta g_\nu}{\delta f_\nu}, \end{array}$$

Let the determinant of the applied substitution (which must not vanish), be denoted by K, the corresponding minors $\dfrac{\delta K}{\delta K_{a,b}}$ by $K_{a,b}$; then the substitution

$$(7)\qquad \begin{array}{l} \dfrac{K_{1,1}}{K},\ \dfrac{K_{1,2}}{K},\ \ldots\ \dfrac{K_{1,n}}{K}, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad, \\ \dfrac{K_{n,1}}{K},\ \dfrac{K_{n,2}}{K},\ \ldots\ \dfrac{K_{n,n}}{K}, \end{array}$$

is said to be *contrary* to the original substitution (5). These things established, Mr. Sylvester gives a general theorem couched in the following terms:

In a prepared form two contrary substitutions operated on the variables induce two contrary substitutions operated on the elements.

Mr. Sylvester having proved this remarkable truth by ascending from binary forms to ternary, from these to quaternary and so on, the present paper will contain a demonstration that embraces the whole theorem in one grasp.

In order to make use of the substitution (7), let the n variables $x_1, x_2, \ldots x_n$ be made equal to the following linear functions of n new variables $z_1, z_2, \ldots z_n$,

$$(8)\qquad x_a = \frac{K_{a,1}}{K} z_1 + \frac{K_{a,2}}{K} z_2 + \ldots + \frac{K_{a,n}}{K} z_n,$$

according to which the form $F(x_1, x_2, \ldots x_n)$ will be changed into the likewise prepared form

$$(9)\qquad H(z_1, z_2, \ldots z_n) = \sqrt{\pi_1} h_1 Z_1 + \sqrt{\pi_2} h_2 Z_2 + \ldots + \sqrt{\pi_\nu} h_\nu Z_\nu,$$

that contains the constants $h_1, h_2, \ldots h_\nu$. Now we may observe that, on solving the system (8) with respect to the quantities z_b, there arise the equations

$$z_b = k_{1,b}x_1 + k_{2,b}x_2 + \ldots + k_{n,b}x_n, \tag{10}$$

where the system of coefficients is got from (5) by interchanging the horizontal and vertical lines with each other, or by transposition. Consequently, the form $H(z_1, z_2, \ldots z_n)$ is transformed into the form $F(x_2, x_2, \ldots, x_n)$ by the substitution

$$\begin{matrix} k_{1,1}, & k_{2,1}, & \ldots & k_{n,1}, \\ \cdot & \cdot & \cdot & \cdot\ , \\ \cdot & \cdot & \cdot & \cdot\ , \\ k_{1,n}, & k_{2,n}, & \ldots & k_{n,n}, \end{matrix} \tag{11}$$

which is the substitution (5) transposed; wherefore, if we form the equations

$$\begin{aligned} f_1 &= \frac{\delta f_1}{\delta h_1} h_1 + \frac{\delta f_1}{\delta h_2} h_2 + \ldots + \frac{\delta f_1}{\delta h_\nu} h_\nu, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\ , \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\ , \\ f_\nu &= \frac{\delta f_\nu}{\delta h_1} h_1 + \frac{\delta f_\nu}{\delta h_2} h_2 + \ldots + \frac{\delta f_\nu}{\delta h_\nu} h_\nu, \end{aligned} \tag{12}$$

it is evident that the coefficient $\frac{\delta f_a}{\delta h_\beta}$ may be formed from the coefficient $\frac{\delta g_a}{\delta f_\beta}$ by changing the substitution (5) into (11), or by changing $k_{a,b}$ into $k_{b,a}$. Considering the effect of the equations (8) upon the form $F(x_1, x_2, \ldots x_n)$ we must say that the substitution (7) operated on the variables induces the substitution operated on the elements

$$\begin{matrix} \frac{\delta h_1}{\delta f_1}, & \frac{\delta h_1}{\delta f_2}, & \ldots & \frac{\delta h_1}{\delta f_\nu}, \\ \cdot & \cdot & \cdot & \cdot\ , \\ \cdot & \cdot & \cdot & \cdot\ , \\ \frac{\delta h_\nu}{\delta f_1}, & \frac{\delta h_\nu}{\delta f_2}, & \ldots & \frac{\delta h_\nu}{\delta f_\nu}, \end{matrix} \tag{13}$$

Therefore, it is to be shown, that the substitution (13) is contrary to the substitution (6). But a general property of partial differential coefficients of n functions taken according to n independent variables teaches us that the substitution contrary to (13) is represented with the aid of the partial differential coefficients of the n variables taken according to the n respective functions, as follows:

(14)
$$\begin{array}{l} \frac{\delta f_1}{\delta h_1}, \frac{\delta f_2}{\delta h_1}, \dots \frac{\delta f_\nu}{\delta h_1}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad , \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad , \\ \frac{\delta f_1}{\delta h_\nu}, \frac{\delta f_2}{\delta h_\nu}, \dots \frac{\delta f_\nu}{\delta h_\nu}. \end{array}$$

Of course our theorem only requires us to establish that the substitutions (6) and (14) accord with each other, or that for any combination of the numbers α, β the equation

(15)
$$\frac{\delta f_\alpha}{\delta h_\beta} = \frac{\delta g_\beta}{\delta f_\alpha}$$

is valid. Meanwhile, we have seen that the partial differential coefficient $\frac{\delta g_\alpha}{\delta f_\beta}$ is changed into $\frac{\delta f_\alpha}{\delta h_\beta}$ by changing $k_{a,b}$ into $k_{b,a}$. Consequently, it will suffice to prove, *that the partial differential coefficient* $\frac{\delta g_\alpha}{\delta f_\beta}$ *turns into the partial differential coefficient* $\frac{\delta g_\beta}{\delta f_\alpha}$, *if* $k_{a,b}$ *is changed into* $k_{b,a}$.

Supposing that the forms in question are expressed in the usual manner,

(16) $$F(x_1, x_2, \dots x_n) = \pi_1 F_1 X_1 + \pi_2 F_2 X_2 + \dots \pi_\nu F_\nu X_\nu,$$

(17) $$G(y_1, y_2, \dots y_n) = \pi_1 G_1 Y_1 + \pi_2 G_2 Y_2 + \dots \pi_\nu G_\nu Y_\nu,$$

(18) $$H(z_1, z_2, \dots z_n) = \pi_1 H_1 Z_1 + \pi_2 H_2 Z_2 + \dots \pi_\nu H_\nu Z_\nu,$$

the constants f_α, g_β, h_γ are connected with the constants F_α, G_β, H_γ, by the purely numerical relations,

(19) $$f_\alpha = \sqrt{\pi_\alpha} F_\alpha, \; g_\beta = \sqrt{\pi_\beta} G_\beta, \; h_\gamma = \sqrt{\pi_\gamma} H_\gamma,$$

so that, instead of (15), we have the equation

(20) $$\pi_\alpha \frac{\delta F_\alpha}{\delta H_\beta} = \pi_\beta \frac{\delta G_\beta}{\delta F_\alpha}.$$

In order to prove the former, we are going to prove the latter. Taking notice of the fact that the partial differential coefficient $\frac{\delta G_\alpha}{\delta F_\beta}$ turns into $\frac{\delta F_\alpha}{\delta H_\beta}$ by changing $k_{a,b}$ into $k_{b,a}$, the meaning of (20) may be expressed in the words, *that the product* $\pi_\alpha \frac{\delta G_\alpha}{\delta F_\beta}$ *is changed into the product* $\pi_\beta \frac{\delta G_\beta}{\delta F_\alpha}$ *by changing* $k_{a,b}$ *into* $k_{b,a}$.

As it is permitted to regard the quantities Y_β as linear functions of the quantities X_α and reciprocally, by differentiating the equation

$$F(x_1, x_2, \dots x_n) = G(y_1, y_2, \dots y_n)$$

first according to F_α, afterwards according to Y_β, we shall find

(21) $$\pi_\alpha X_\alpha = \sum_\beta \pi_\beta \frac{\delta G_\beta}{\delta F_\alpha} Y_\beta; \quad \pi_\alpha \frac{\delta X_\alpha}{\delta Y_\beta} = \pi_\beta \frac{\delta G_\beta}{\delta F_\alpha}.$$

In like manner from the equation $F(x_1, x_2, \ldots x_n) = H(z_1, z_2, \ldots z_n)$ result the equations

(22) $$\sum_\alpha \pi_\alpha \frac{\delta F_\alpha}{\delta H_\beta} X_\alpha = \pi_\beta Z_\beta; \quad \pi_\alpha \frac{\delta F_\alpha}{\delta H_\beta} = \pi_\beta \frac{\delta Z_\beta}{\delta X_\alpha}.$$

Whence it is evident, that the equation (20) will be proved true if the equation

(23) $$\pi_\alpha \frac{\delta X_\alpha}{\delta Y_\beta} = \pi_\beta \frac{\delta Z_\beta}{\delta X_\alpha}$$

is proved to hold good.

This equation containing no trace of the respective forms, let us denote by $t_1, t_2, \ldots t_n$ a set of n new independent variables, with which we form the expression

(24) $$t_1 x_1 + t_2 x_2 + \ldots + t_n x_n,$$

which is to be elevated to the pth power. By the aid of the equation (2) and of the definition

(25) $$u_b = k_{1,b} t_1 + k_{2,b} t_2 + \ldots + k_{n,b} t_n$$

the expression (24) assumes the following shape

(26) $$u_1 y_1 + u_2 y_2 + \ldots + u_n y_n.$$

Hence arise by means of the previously introduced notation the expressions

(27) $$(t_1 x_1 + t_2 x_2 + \ldots t_n x_n)^p = \pi_1 T_1 X_1 + \pi_2 T_2 X_2 + \ldots + \pi_\nu T_\nu X_\nu,$$

(28) $$(u_1 y_1 + u_2 y_2 + \ldots + u_n y_n)^p = \pi_1 U_1 Y_1 + \pi_2 U_2 Y_2 + \ldots + \pi_\nu U_\nu Y_\nu.$$

Considering that X_α and Y_β as well as T_α and U_β depend linearly on one another, we may differentiate the equivalent expressions (27) and (28), first according to T_α, afterwards to Y_β, and get

(29) $$\pi_\alpha X_\alpha = \sum_\beta \pi_\beta \frac{\delta U_\beta}{\delta T_\alpha} Y_\beta; \quad \pi_\alpha \frac{\delta X_\alpha}{\delta Y_\beta} = \pi_\beta \frac{\delta U_\beta}{\delta T_\alpha}.$$

But as by virtue of (25) and (10) the variables u_b depend upon the variables t_a in the same way as the variables z_b depend upon the variables x_a, we conclude that the quantities U_β must be the same linear functions of the quantities T_α as the quantities Z_β are of the quantities X_α, and that consequently the partial differential coefficients $\frac{\delta U_\beta}{\delta T_\alpha}$ and $\frac{\delta Z_\beta}{\delta X_\alpha}$ denote the same thing. Hence it follows that the second equation in (29) produces the equation (23) as was to be proved, and thus the demonstration of the proposed theorem is accomplished.